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MULTIOBJECTIVE OPTIMIZATION OF LOW-ENERGY TRAJECTORIES USING OPTIMAL CONTROL ON DYNAMICAL CHANNELS

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We introduce a computational method to design efficient low-energy trajectories by extracting initial solutions from dynamical channels formed by invariant manifolds, and improving these solutions through variational optimal control. We consider trajectories connecting two unstable periodic orbits in the circular restricted 3-body problem (CR3BP). Our method leverages dynamical channels to generate a range of solutions, and approximates the Pareto front for impulse and time of flight through a multiobjective optimization of these solutions based on primer vector theory. We demonstrate the application of our method to a libration orbit transfer in the Earth-Moon system.

INTRODUCTION

Orbit transfers accomplished using low-energy trajectories frequently exhibit significant trade-offs between the required propulsive impulse (ΔV) and required time of flight (TOF). Mission designers can benefit greatly from knowledge of a range of efficient (non-dominated) options in the multiobjective minimization of ΔV and TOF. The highly nonlinear dynamics that enable low-energy trajectories in multi-body gravitational models also pose challenges to identifying such a range of efficient options. The method described here generates candidate solutions from dynamical channels formed by intersections of invariant manifolds of the initial and target orbits in the circular restricted 3-body problem (CR3BP), and locally improves these solutions through variational optimal control methods to construct an approximation to the Pareto front of efficient options.

Numerous studies have demonstrated the utility of dynamical channels[1] as a tool for finding efficient trajectories in the unstable nonlinear dynamical regions of the CR3BP. In particular, certain pairs of libration orbits exhibit transfer trajectories with no deterministic maneuvers,[1–3] and many pairs of mean motion resonance orbits show evidence of transfer trajectories requiring only small propulsive maneuvers.[4,5] Moreover, examinations of multiple efficient trajectories found by extensive search and numerical optimization indicate a close correspondence with nearby dynamical channels.[6–10;11]

Several methods for trajectory design have employed invariant manifolds to find efficient solutions. Early work explored transfers to libration point orbits using their stable manifolds,[12] and applied Floquet theory to optimize transfers between halo orbits of different energies in the same family.[13] For trajectories to libration point orbits, research has explored construction and optimization of transfers targeting stable manifolds, using differential correction and constrained gradient optimization,[14] thrust-limited tangent steering laws,[15,16] and thrust-limited optimization by both direct[17,18] and indirect[19] methods. To transfer between arbitrary unstable periodic orbits connected by dynamical channels, several studies have used Poincaré sections to identify desirable channels,[20–25] and others have applied direct optimization methods to refine such solutions.[26,27] Other investigations for particular mission types have parameterized limited families of impulsive dynamical-channel transfers with a small set of variables subject to numerical optimization.[28,29] Another recent approach constructs transfers between invariant manifold segments selected according to a two-body heuristic criterion, which can be likewise optimized according to a small number of parameters.[30]

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Our method leverages dynamical channels to explore the space of efficient low-energy solutions across a wide range of the objective parameters. We introduce a global optimization procedure based on numerical continuation to search for intersecting segments of invariant manifolds propagated up to a desired TOF bound. In examples of interest, the unstable and stable manifolds of initial and target orbits can form multiple disconnected dynamical channels that offer different trade-offs between ΔV and TOF.[1,4,20,31,32] Extracting initial solutions from multiple such channels yields a dynamically motivated sample of trajectories distributed across a region expected to lie near the ΔV-TOF Pareto front. The local multiobjective optimization of these trajectories then yields an approximation to the Pareto front within the desired bounds.

For local optimization, we initially employ a fixed-TOF formulation of the indirect primer vector approach to optimal control, originally developed by Lawden[33] for two-body trajectory optimization. Prior investigations have applied this approach to both impulsive[34,35] and thrust-limited[36–38] trajectories with three-body dynamics. We follow the general strategy that the Apollo-era research of Lion & Handelsman[39] and Jezewski & Rozendaal[40] developed to construct locally optimal impulsive transfers in the two-body problem. This recursive procedure employs the time profile of the primer vector on a suboptimal trajectory to guide the differential adjustment or addition of impulsive maneuvers to satisfy the necessary optimality conditions.

Starting from locally optimal fixed-TOF trajectories corresponding to the initial solutions generated from dynamical channels, we construct nearby locally optimal trajectories by varying the TOF parameter while retaining the desired optimality conditions. In the immediate neighborhood of the starting solutions, this is accomplished using a variational form of the optimality conditions to guide the iterative computation of nearby solutions. Outside an immediate neighborhood of each starting solution, further iterations of the recursive procedure described above are necessary to retain local optimality; the requisite condition is the same used by the original procedure to determine the adjustment of impulses based on the primer vector profile. From each starting solution, we generate a sufficient set of nearby trajectories to fill the gaps between starting solutions, and thus obtain an effective approximation to the desired Pareto front.

We demonstrate the application of our method to find efficient options for a libration point orbit transfer in the Earth-Moon system. For clarity of illustration, we consider a planar transfer between Lyapunov orbits about the L_1 and L_2 libration points, though the method is not limited to planar transfers. These two orbits have differing values of Jacobi's integral, so that no propulsion-free transfer exists between them in the CR3BP. However, our method discovers a range of locally optimal impulsive transfer solutions that describe the trade-off between ΔV and TOF.

METHODS

Our approach breaks down the search for efficient transfer trajectories into two stages: First, we construct a set of approximate multiobjective minimizers of the transfer problem, each representing a dynamical channel connecting the initial and target orbits (§). These solutions are found through a global optimization approach based on numerical continuation (§). Second, we improve these solutions by an iterative process based on local optimality conditions formulated in terms of primer vector theory (§). The resulting locally optimized solutions are taken to approximate a portion of the globally efficient solution set.

Constructing Transfers from Dynamical Channels

The *circular restricted 3-body problem* (CR3BP) models the mutual gravitation of three bodies as point masses, where the third body has negligible mass, and the primary and secondary bodies move in a circular orbit about their gravitational barycenter. In applications, the third body typically represents a spacecraft or small object, while the primary and secondary bodies typically represent a star-planet or planet-moon pair. We choose a system of units and synodic rotating coordinates x, y, z so that:[41] the secondary body has mass μ and the primary has mass $1-\mu$; the primary is fixed at coordinates $(-\mu, 0)$ and the secondary at $(1-\mu, 0)$; and the equations of the third body's motion are given by

$$
\begin{aligned}
\ddot{x} - 2\dot{y} &= \partial_x \Omega & (1)\\
\ddot{y} + 2\dot{x} &= \partial_y \Omega & (2)\\
\ddot{z} &= \partial_z \Omega, & (3)
\end{aligned}
$$

where $\partial_x \Omega$, $\partial_y \Omega$, $\partial_z \Omega$ denote the derivatives with respect to x, y, z of

$$\Omega(x, y, z) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}, \tag{4}$$

with distances from the primary and secondary massive bodies to the third body given by

$$
\begin{aligned}
r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2} & (5)\\
r_2 &= \sqrt{(x-1+\mu)^2 + y^2 + z^2}, & (6)
\end{aligned}
$$

respectively. The *planar* CR3BP (PCR3BP) is the restriction of the CR3BP to the plane $z = 0$.

In the above coordinates, the CR3BP has a first integral (conserved quantity) traditionally expressed as the *Jacobi integral*

$$C = 2\Omega - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{7}$$

The existence of the Jacobi integral implies that any trajectory of the third body lies on a 5-dimensional manifold imbedded in 6-dimensional phase space. In the planar problem, each trajectory lies on a 3-dimensional manifold imbedded in 4-dimensional phase space. Any transfer between orbits with different values of Jacobi's integral requires some propulsive impulse.

Unstable periodic orbits in the CR3BP have 2-dimensional *invariant manifolds* constituting those trajectories that asymptotically converge to (*stable* manifold) and diverge from (*unstable* manifold) the periodic orbit. The invariant manifolds of a periodic orbit represent all states reachable from the periodic orbit moving backward and forward (respectively) in time with no propulsive maneuvers. When the stable manifold of one periodic orbit intersects the unstable manifold of another periodic orbit, we term this connection a *dynamical channel* from the second orbit to the first. If the two manifolds intersect in position space only, a trajectory from one orbit to the another may be accomplished by a single propulsive maneuver at the intersection point, cancelling the difference in velocity. If the two manifolds intersect in the full state space, there exists a corresponding trajectory with no propulsive maneuvers.

Generating Solutions by Numerical Continuation

We construct candidate transfer solutions by searching for dynamical channels formed by the stable manifold of the target orbit and the unstable manifold of the initial orbit. Describing each invariant manifold by two free parameters, this search may be initially formulated as a four-dimensional global minimization of the position-space distance between points on the manifolds (for smoothness, we use the squared Euclidean norm). The highly nonlinear geometry of the invariant manifolds motivates a specialized approach to this global optimization based on numerical continuation.

Invariant manifolds of a periodic orbit in the CR3BP have no effective global representation, either explicit or implicit: rather, they are locally defined in relation to the orbit by the equations of motion. The manifolds of an orbit can be approximated near the orbit by the linearized dynamics around the orbit, expressed by the monodromy matrix. The eigendecomposition of this matrix indicates the local linear subspace tangent to the manifold. The unstable subspace correspond to eigenvalues (Floquet multipliers) greater than one. Stable manifolds may be identified as unstable manifolds of periodic orbits reversed in time. One can compute a numerical approximation to the manifold by numerical integration of points in the local tangent space slightly offset from the periodic orbit.

Some efficiencies in the implementation of our approach depend upon an advantageous choice of parameters to describe each invariant manifold. We generate the manifold from points lying on a linear segment of the (one-dimensional) local tangent space near a given point on the periodic orbit, such that the two ends of the segment are approximately connected by a trajectory near the manifold that winds around the manifold exactly once. This ensures that the propagation of the entire segment constitutes an approximation to the entire manifold, provided the entire segment is sufficiently close to the periodic orbit. When comparing times of flight between two trajectory segments on the same manifold, the difference in offset is accounted for using the linear approximation to the dynamics near the periodic orbit.

The two parameters describing each manifold approximation are thus the offset ϵ of the starting point along the generating segment, and the integration time T with which the starting point is propagated. Denoting the unstable manifold of the initial orbit by the superscript ^{u} and the stable manifold of the target orbit by the superscript ^{s}, the set of combined approximate manifold segments is mapped by the four parameters ϵ^u, T^u, ϵ^s, and T^s. Variations in T^u and T^s correspond to numerical integration of manifold segments in time, while variations in ϵ^u and ϵ^s correspond to smooth variations of the segments in space.

The existence of a dynamical channel requires that the position components of the corresponding endpoints of the two manifold segments are equal, that is, their difference is zero. Hence the set of such points is the set of global minimizers of this difference in the four-parameter space describing the manifold segments, and the search for dynamcial channels can be cast as a multivariate optimization problem in the residual. For smoothness, we actually consider its squared Euclidean norm as the objective in the optimization.

Invariant manifolds in the CR3BP typically exhibit very large variations in curvature with respect to both ϵ and T, which can lead to poor results for many common approaches to global optimization. Moreover, since generic global optimization methods are generally constructed based on isolated independent function evaluations, they fail to allow for the fact that the invariant manifolds are not globally defined, leading to computational inefficiencies. We thus propose a specialized optimization approach based on numerical continuation that leverages the special structure of the problem and addresses its inherent computational challenges.

Given a solution to a system of ordinary differential equations with state variables u and scalar parameters λ, and boundary conditions expressed in terms of these variables, consider the goal of describing the family of solutions that results as one of the parameters λ_i is varied. In our approach, the system of differential equations is discretized by adaptive Gaussian orthogonal collocation to form an algebraic system approximating the solution to the boundary value problem. The method of pseudo-arclength continuation[?] is used to vary the solution according to the parameter λ_i: variables are incremented to as to approximate a constant step size in the collocation's combined extended variable-parameter space. Monitoring the Jacobians of the collocation system allows the solution to detect and continue through folds (extrema) with respect to the parameter λ_i.

As the solution is continued in λ_i, the collocation mesh is adapted according to the curvature of the local solution. In our application, the pseudo-arclength parameterization adapts manifold segments to changing curvature with respect to ϵ, and the mesh refinement adapts the segments to changing curvature with respect to T, as either parameter is varied. (The latter, combined with sparse linear solvers, may be considered to provide advantages analogous to those of variable step size numerical integration.) This approach thus provides a globally adaptive representation of the segment solution sets for each manifold, addressing the principal computational difficulties associated with these objects. Since each manifold's parameters are independent of each other, intelligent caching of intermediate results can allow relatively fast traversals throughout the parameter space.

It is well known that multivariate local extrema can be isolated by the method of *successive continuation*:[?] here a fold is first located with respect to parameter λ_1; an extended system is formed to continue this fold with respect to the parameter λ_2, simultaneously varying λ_1; and so on, until finding a fold with respect to all the parameters λ. However, this method generally fails for global optimization: in particular, while continuation in a single parameter traverses an unbounded sequence of folds, simultaneous continuation of folds in multiple parameters will often traverse a closed loop in parameter space containing only a small number of higher-order folds.

Our approach to global optimization uses continuation of folds to leverage the information about local curvature implicitly provided by the continuation process, thus adapting the search to the complicated geometric structure of the manifolds; however, it applies a multi-layered form of successive continuation in order to ensure a comprehensive exploration of the parameter space. Beginning from a starting point (ideally near the center of the parameter space of interest), the procedure generates candidate multivariate minimizers through a recursive process.

First, each parameter λ_i is assigned an integer layer bound b_i; these are used to limit the recursion depth. For the starting solution, each parameter λ_i is assigned an initial layer count $c_i = 0$. For each parameter λ_i, the solution is then continued in both directions with respect to λ_i alone, within the extent of the desired parameter space, locating folds in the objective function relative to λ_i; the resulting fold points are added to the solutions set with $c_i = 1$. For each solution in the solution set, the process is recursively repeated, continuing and locating folds independently in each λ_i (except the immediately preceding one), and incrementing the corresponding layer count c_i. When the layer count c_i reaches the layer bound b_i, instead of continuing with respect to subsequent parameters independently, the fold located with respect to λ_i is continued in multiple parameters, thus following multivariate folds. In the end, all solutions are multivariate folds in all the λ_i, having been effectively locally optimized by successive continuation.

This layered process adapts to the changing curvature of the solution space by everywhere following the local structure of folds in the objective function, and branching to cover the space before closing in on local extrema. Intuitively, folds in the objective with respect to λ_i at a given point in parameter space separate intervals in which the objective is monotonic in λ_i; hence sampling on fold points effectively adapts to the local smoothness of the objective function. Given particular layer bounds b_i, all local extrema are located that are reachable by following a sequence of folds from the starting point (including multivariate folds in the final layer) that contain b_i instances of λ_i. The necessary layer bounds depend on the difficulty of the problem and the quality of results desired; for simple examples, we have found $b_i = 2$ for all parameters to be reasonably effective.

The complexity of the method may be loosely considered multiply exponential in the density of folds with respect to each parameter λ_i, with the multipliers b_i; however, since the density of folds varies across the parameter space, this characterization can be misleading. In the end, for given parameters, the method in some sense adapts to the true complexity of the problem, with the bounds b_i providing a means to temper unnecessary refinement. In addition, the method is easily parallelized, and the space complexity of the generated solution set can be mitigated by detecting duplicate local extrema as they are located. Our implementation makes use of the software package AUTO for continuation and bifurcation analysis of ordinary differential equations.[?]

For our purposes, only solutions with zero objective value (position-space intersections of manifold segments) are considered, as they constitute true dynamical channels. Future work will investigate the possibility of constructing candidate transfer solutions from near-intersections using multi-impulse adjoining arcs.

Improving Solutions by Optimal Control

To locally optimize candidate solutions generated by dynamical channels, we follow the general strategy developed in the work of Lion & Handelsman[39] and Jezewski & Rozendaal.[40] Prior research has used a similar strategy to locally optimize transfer trajectories connecting points on manifolds;[35] in our approach, we apply the optimization to the complete candidate solution generated from each dynamical channel, with endpoints at the corresponding offsets from the initial and target periodic orbits, and an interior impulse at the intersection of the manifold segments.

The primer vector refers to the vector of Lagrange multipliers corresponding to the velocity vector in a standard optimal control formulation. A set of local optimality conditions on the trajectory may be stated in terms of the primer vector:[33]

1. The primer vector and first derivative of the primer vector are continuous along the entire trajectory;
2. At each propulsive impulse, the primer vector lies in the direction of the impulse with unit magnitude;

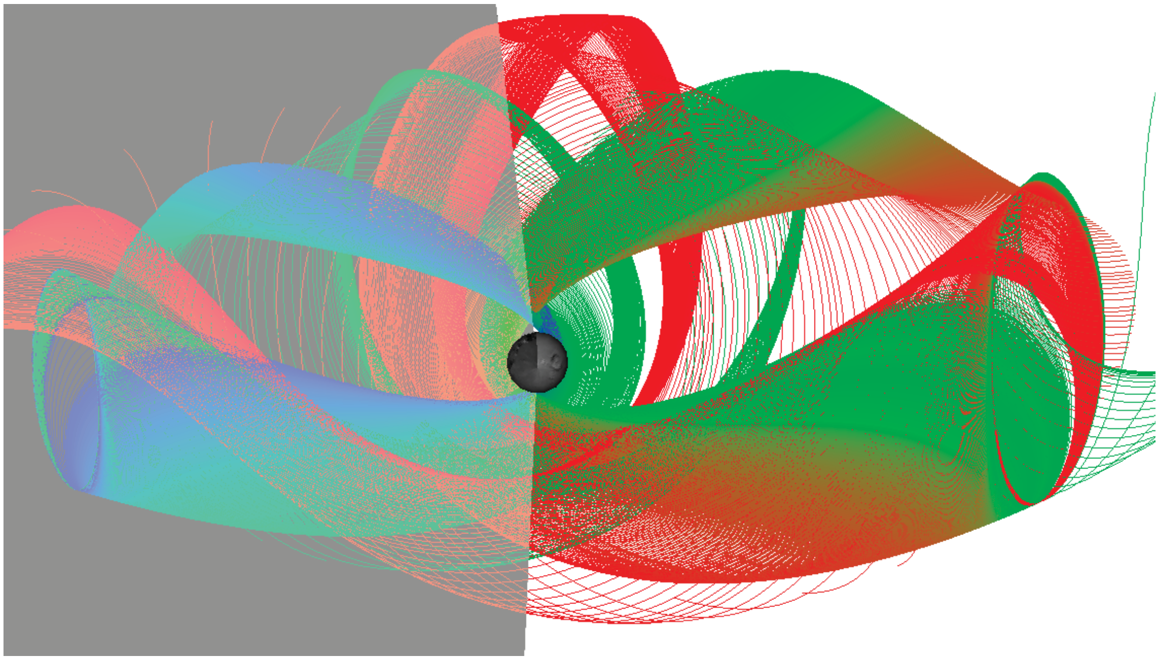

Figure 1: Effective sampling of invariant manifolds achieved by numerical continuation

3. Everywhere else on the trajectory, the primer vector magnitude does not exceed unity;
4. At each interior impulse, the first derivative of the magnitude of the primer vector is zero.

Given a reference solution, if the primer vector magnitude anywhere exceeds unity, the maximum improvement, to first order, in the trajectory can be achieved by adding an interior impulse at the point at which the primer vector magnitude is maximum, in the direction of the primer vector. The magnitude and timing of the impulse can be subsequently adjusted by iterative root-finding in a small number of parameters to satisfy the optimality conditions. The above procedure may then be iterated to produce a trajectory satisfying the local optimality conditions.

RESULTS

As an illustrative example, we consider a planar transfer between two Lyapunov libration orbits about the L_1 and L_2 Lagrange points (respectively) of the Earth-Moon system. These Lyapunov orbits are representative of orbits considered for libration point mission applications, and offer convenient transfers to orbits of interest in the Earth and Moon neighborhoods. The initial orbit is an L_1 Lyapunov orbit with Jacobi constant $C = 3.196$, while the target orbit is an L_2 Lyapunov orbit with Jacobi constant $C = 3.178$. As these two constants are unequal, no nonpropulsive transfer exists between these two orbits.

Figure 1 shows portions of the unstable manifold of the L_1 orbit and stable manifold of the L_2 orbit in the region of interest, each propagated out to roughly 4 weeks in duration. As it turns out, all efficient transfers within this window have substantially lower time of flight; however, the geometry is illustrative of the geometric complexity that may prove relevant in other transfer problems. The manifold segments shown in Figure 1 indicate the adaptive sampling effectively achieved by the numerical continuation procedure.

Figure 2 makes visible the Pareto front of objective values achieved by the locally optimal trajectories generated from our method. Some examples from the Pareto front are plotted in Figure 3. Several distinct families of dynamical channels are evident, though the efficient trajectories with the lowest range of ΔV (for instance, Figure 3(a) and (b)) all belong to a single family. A few isolated intersections allow transfers with substantially lower TOF at a cost of greater ΔV (Figure 3(c) and (d)); these have a similar geometry,

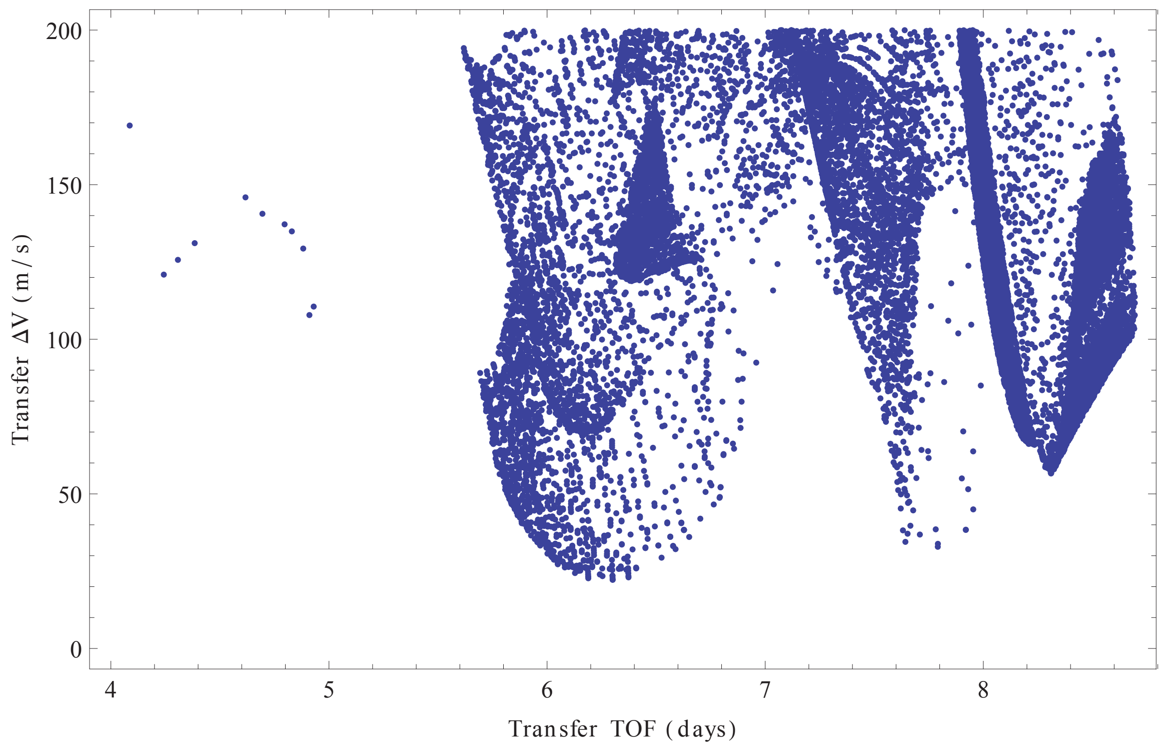

Figure 2: Portion of locally optimal transfer solutions generated

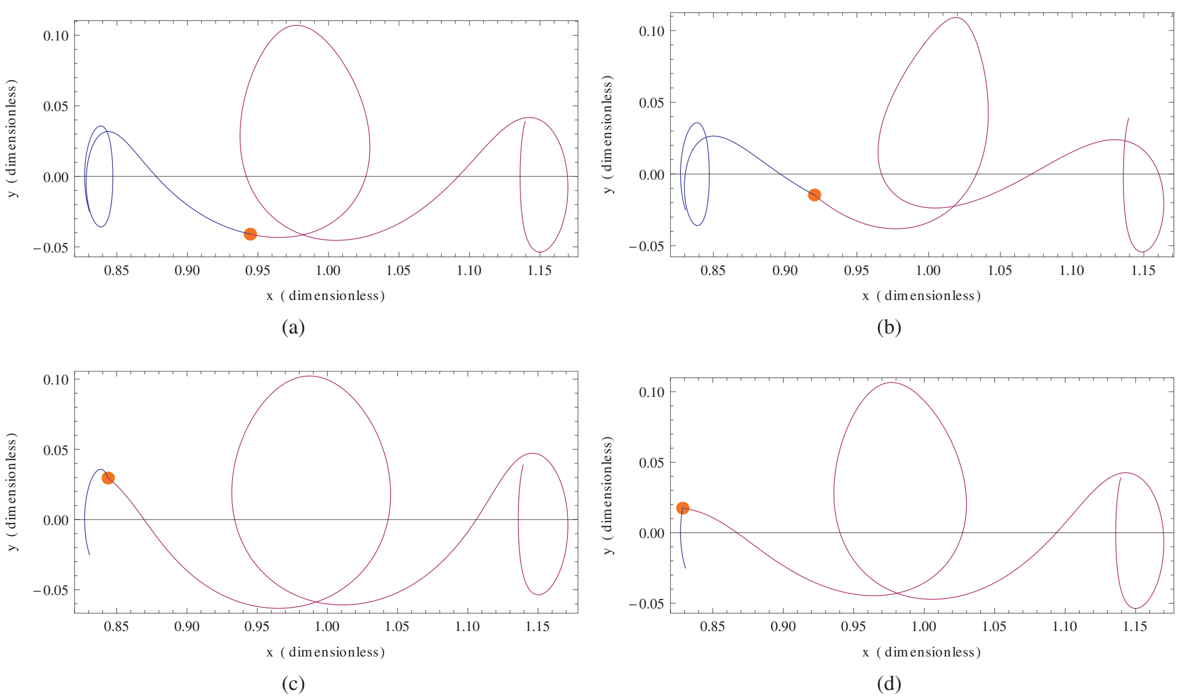

Figure 3: (a-d) Sample efficient trajectories from the Pareto front with increasing ΔV and decreasing TOF

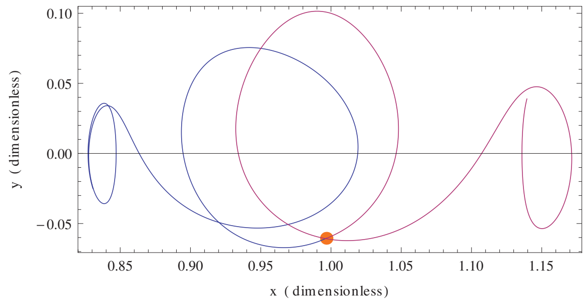

Figure 4: An example trajectory (not Pareto-optimal) generated from different branch of dynamical channels

but transfer earlier to the faster-moving stable manifold of the target orbit. Other families of dynamical channels produce transfers with substantially different geometry: for instance, Figure 4. While not efficient in this example, other problems may exhibit contributions to the Pareto front from many such families; in this situation, a global optimization approach like the one described here can be essential to developing a comprehensive set of options for mission design.

CONCLUSION

We have described a two-stage method for constructing comprehensive multiobjective-optimal solutions to low-energy orbit transfer problems, an important capability to support trajectory design for missions leveraging multi-body dynamics to reduce fuel requirements. A specialized adaptive approach to global optimization is used to identify position-space intersections of invariant manifolds forming dynamical channels, from which candidate solutions may be locally improved using optimal control to generate an approximation to the Pareto front over parameter ranges of interest. The method is demonstrated to produce a considerable range of potentially useful options in an example libration point orbit transfer application, and is easily extensible to more complex problems involving any type of unstable periodic orbits.

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